Quantum Field Theory: Exercise Session 1 26 October 2015

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Problem 1: Poincaré algebra for the real scalar field

The real Klein-Gordon field $\phi(x)$ is governed by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi - \frac{m^2}{2}\,\phi^2. \tag{1}$$

We can write $\phi(x)$ and its conjugate momentum $\pi(x)$ in terms of the creation and annihilation operators $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$ as

$$\phi(\vec{x}) \;=\; \int \tilde{\mathrm{d}}k \left[a_{\vec{k}}\,\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}} + a^\dagger_{\vec{k}}\,\mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{x}}\right], \tag{2}$$

$$\pi(\vec{x}) \;=\; -\mathrm{i}\int \tilde{\mathrm{d}}k\,\omega_k \left[a_{\vec{k}}\,\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}} - a^\dagger_{\vec{k}}\,\mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{x}}\right], \tag{3}$$

where

$$\tilde{\mathrm{d}}k \equiv \frac{\mathrm{d}^3k}{(2\pi)^3 2\omega_k}, \qquad \omega_k \equiv \sqrt{\vec{k}^2 + m^2}, \qquad \vec{k}\cdot\vec{x} \equiv k^i x^j \delta_{ij}. \tag{4}$$

In the lecture, you have seen that energy and momentum may be written in terms of $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$ as

$$\begin{aligned} P^0 \equiv H \equiv \int \mathrm{d}^3x\, T^{00} \;&=\; \int \mathrm{d}^3x\,\frac{1}{2}\left(\pi^2 + \left(\vec{\nabla}\phi\right)^2 + m^2\phi^2\right) \\ &=\; \int \tilde{\mathrm{d}}k\,\omega_k\,a^\dagger_{\vec{k}}\,a_{\vec{k}}, \end{aligned} \tag{5}$$

and

$$P^i \equiv \int \mathrm{d}^3x\,T^{0i} = \int \mathrm{d}^3x\,\pi(x)\nabla^i\phi(x) = \int \tilde{\mathrm{d}}k\,k^i\,a^\dagger_{\vec{k}}\,a_{\vec{k}}, \tag{6}$$

where $T^{\mu\nu}$ is the energy-momentum tensor. The Lorentz generators are given by

$$M^{\mu\nu} \equiv \int \mathrm{d}^3x\,\left(x^\mu T^{0\nu} - x^\nu T^{0\mu}\right). \tag{7}$$

(a) Use the above expressions to write down the boost and rotation generators M_{i0} and M_{ij} in terms of $\phi(x)$ and $\pi(x)$.

(b) Use the Fourier expansions of $\phi(x)$ and $\pi(x)$ to express the rotation generators M_{ij} in terms of $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$.

Hint:

$$\int \mathrm{d}^3x\, x_i\, \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}} = \int \mathrm{d}^3x \left(-\mathrm{i}\frac{\partial}{\partial k_i}\right) \mathrm{e}^{\mathrm{i}\vec{k}.\vec{x}} = -\mathrm{i}(2\pi)^3 \frac{\partial}{\partial k_i}\delta^3(\vec{k}). \tag{8}$$

(c) Compute the commutators $[P^i, \phi(x)]$ and $[M^{ij}, \phi(x)]$ in terms of $\phi(x)$, with the help of the commutator relations for $a_{\vec{k}}$ and $a^\dagger_{\vec{k}}$.

(d) *Optional.* Repeat exercises (b) and (c) for the boost generators M_{0i} to find

$$M_{i0} = -\mathrm{i}\int \tilde{\mathrm{d}}k \omega_k a^\dagger_{\vec{k}} \partial_{k_i} a_{\vec{k}} \tag{9}$$

and compute the commmutator $[M^{i0}, \phi(x)]$. Check that the commutator of M_{i0} with M_{j0} satisfies the Lorentz algebra

$$\left[M^{\sigma\tau}, M^{\alpha\beta}\right] = \mathrm{i}\left(\eta^{\tau\alpha}M^{\sigma\beta} + \eta^{\sigma\beta}M^{\tau\alpha} - \eta^{\sigma\alpha}M^{\tau\beta} - \eta^{\tau\beta}M^{\sigma\alpha}\right). \tag{10}$$

Problem 2: The complex scalar field

The free complex Klein-Gordon scalar field is governed by the Lagrangian density

$$\mathcal{L} = \partial_\mu\phi^\dagger\, \partial^\mu\phi - m^2\, \phi^\dagger\phi. \tag{11}$$

Since the complex scalar field carries two degrees of freedom, quantizing it gives rise to two independent creation operators. The mode expansion for ϕ is

$$\phi(\vec{x}) = \int \tilde{\mathrm{d}}k \left[a_{\vec{k}}\, \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}} + b^\dagger_{\vec{k}}\, \mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{x}}\right], \tag{12}$$

where the operators $a_{\vec{k}}$ and $b_{\vec{k}}$ satisfy the commutation relations:

$$\left[a_{\vec{k}}, a^\dagger_{\vec{p}}\right] = \left[b_{\vec{k}}, b^\dagger_{\vec{p}}\right] = \tilde{\delta}(\vec{k} - \vec{p}) \tag{13}$$

with all other commutators vanishing. The creation operators $a^\dagger_{\vec{k}}$ and $b^\dagger_{\vec{k}}$ create two types of particle, both of mass m and spin zero, which are interpreted as particles and anti-particles.

Notice that $\mathcal{L}$ is invariant under the rigid phase transformation $\phi \to e^{i\alpha}\phi$. Associated to this symmetry, Noether's theorem gives the conserved charge

$$Q = \mathrm{i}\int \mathrm{d}^3x \left(\partial_0\phi^\dagger\phi - \phi^\dagger\partial_0\phi\right). \tag{14}$$

(a) Write down the mode expansion for $\phi^\dagger$ and the conjugate momenta, $\pi, \pi^\dagger$.

(b) Express H and Q in terms of the creation and annihilation operators. Show that $[H, Q] = 0$ and give the interpretation of Q. Comment also on the implications that the theory is free.